Vacuum polarization screening corrections to the ground state energy of two-electron ions

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Abstract

Vacuum polarization screening corrections to the ground state energy of two-electron ions are calcualated in the range $Z = 20 - 100$. The calculations are carried out for a finite nucleus charge distribution.

1 Introduction

It is well known that the dominant contribution to the vacuum polarization correction in a strong Coulomb field arises from the Uehling potential [1]. The remaining part of this correction is called by the Wichman-Kroll contribution [2]. Calculations of the Uehling contribution cause no problem and have been done by many authors. Accurate calculations of the Wichman-Kroll contribution were carried out in [3, 4, 5].

Recent measurements of the Lamb shift in highly charged ions [6, 7] have shown a necessity of calculation of vacuum polarization screening diagrams, i.e. diagrams with one vacuum loop and one electron-electron interaction (Fig.1). In the present paper an accurate calculation of these diagrams for the ground state of a two-electron ion is given. Results of the calculations are compared with a previous evaluation of this correction [8].

Relativistic units ($\hbar = c = 1$) are used in the paper.

2 Basic formulas

The vacuum polarization screening diagrams are shown in Fig.1. The formal expressions for the energy level shift due to these diagrams can easily be

derived using the two-time Green function method [9]. The contribution of the diagrams shown in Fig 1a is

$$\Delta E_a = \sum_P (-1)^P \sum_{\varepsilon_n \neq \varepsilon_a} \{ \langle PaPb \mid I(0) \mid nb \rangle \frac{1}{\varepsilon_a - \varepsilon_n} \langle n \mid U^a_{VP} \mid a \rangle + \langle PaPb \mid I(0) \mid an \rangle \frac{1}{\varepsilon_a - \varepsilon_n} \langle n \mid U^a_{VP} \mid b \rangle + \langle Pa \mid U^a_{VP} \mid n \rangle \frac{1}{\varepsilon_a - \varepsilon_n} \langle nPb \mid I(0) \mid ab \rangle + \langle Pb \mid U^a_{VP} \mid n \rangle \frac{1}{\varepsilon_a - \varepsilon_n} \langle Pan \mid I(0) \mid ab \rangle \}. \tag{1}$$

Here U^a_{VP} is the vacuum polarization potential:

$$U^a_{VP}(\mathbf{x}) = \frac{\alpha}{2\pi i} \int d\mathbf{y} \frac{1}{|\mathbf{x}-\mathbf{y}|} \int_{-\infty}^{\infty} d\omega Tr(G(\omega, \mathbf{y}, \mathbf{y})), \tag{2}$$

$$I(\omega, \mid \mathbf{x}-\mathbf{y} \mid) = \alpha \frac{\alpha_{1\mu}\alpha_2^{\mu}}{\mid \mathbf{x}-\mathbf{y} \mid} \exp(i \mid \omega \mid\mid \mathbf{x}-\mathbf{y} \mid), \tag{3}$$

$\alpha^\mu \equiv (1, \boldsymbol{\alpha})$, $\boldsymbol{\alpha}$ are the Dirac matrices; a, b are the $1s$ states with a spin projection $m = \pm\frac{1}{2}$; P is the permutation operator; $G(\omega, \mathbf{x}, \mathbf{y}) = \sum_n \frac{\psi(\mathbf{x})\psi^\dagger(\mathbf{y})}{\omega - \varepsilon_n(1-i0)}$ is the Coulomb Green function.

The contribution of the diagrams shown in Fig 1b is

$$\Delta E_b = \sum_P (-1)^P \langle PaPb \mid U^b_{VP} \mid ab \rangle, \tag{4}$$

where

$$U^b_{VP}(\mathbf{x}, \mathbf{y}) = \frac{\alpha^2}{2\pi i} \int_{-\infty}^{\infty} d\omega \int d\mathbf{z}_1 \int d\mathbf{z}_2 \frac{\alpha_{1\mu}}{|\mathbf{x}-\mathbf{z}_1|} \frac{\alpha_{2\nu}}{|\mathbf{y}-\mathbf{z}_2|} \times Tr(\alpha^\mu G(\omega, \mathbf{z}_1, \mathbf{z}_2) \alpha^\nu G(\omega, \mathbf{z}_2, \mathbf{z}_1)). \tag{5}$$

The contributions (1) and (4) are ultraviolet divergent. The most simple way to renormalize these contributions is to expand the vacuum loop in powers of

the external field (αZ). In this case the contribution of the diagrams with an odd number of vertices in the vacuum loop (with free-electron propagators) is equal to zero according to the Furry theorem. In this expansion, only the first nonzero term, called as the Uehling term, is infinite. The charge renormalization makes this term finite and its calculation causes no problem. The higher orders (in αZ) terms are finite. However, regularization is still needed in the second nonzero term due to the spurious gauge dependent piece of the light-by-light scattering contribution. As it was shown in [10, 11, 3], in the calculation of the vacuum polarization charge density, based on the partial wave expansion of the electron Green function, the spurious term does not contribute if the sum over the angular momentum quantum number κ is restricted to a finite number of terms ($|\kappa| \leq K$). We found that this rule is also correct in the case of the diagrams shown in Fig.1b (see Appendix A). Thus the Wichman-Kroll contribution is calculated by summing up the partial differences between the full contribution and the Uehling term.

3 Calculation

3.1 Diagrams "a"

Renormalized expression for the Uehling potential is well known

$$\begin{aligned} U^a_{Uehl}(r) &= -\alpha Z \frac{2\alpha}{3\pi} \int\limits_0^\infty dr' 4\pi r' \rho(r') \int\limits_1^\infty dt \left(1 + \frac{1}{2t^2}\right) \frac{\sqrt{t^2-1}}{t^2} \\ &\quad \times \frac{[\exp(-2m|r-r'|t) - \exp(-2m(r+r')t)]}{4mrt}, \end{aligned} \quad (6)$$

where $|e|Z\rho(r)$ is the density of the nucleus charge distribution ($\int \rho(r) d\mathbf{r} = 1$). It is interesting to note that with a good precision ($\sim 0.3\%$ for $Z = 80$) the formula (6) can be replaced by a simpler formula (see Appendix B):

$$U^a_{Uehl}(r) \approx V(r) \frac{2\alpha}{3\pi} \int\limits_1^\infty dt \left(1 + \frac{1}{2t^2}\right) \frac{\sqrt{t^2-1}}{t^2} \exp(-2mrt), \quad (7)$$

where $V(r)$ is the potential of an extended nucleus. The calculations were carried out for the Fermi model of the nuclear charge distribution using the

exact formula (6). To calculate the reduced Green function which appears in the formula (1) the B-spline method for the Dirac equation [12] was used. A change of the result, due to a one percent variation of the root-mean-square charge radius of the nucleus, was chosen as the uncertainty.

The calculation of the Wichman-Kroll contribution caused no problem too. Rotating the contour of the ω integration in the complex ω plane along the imaginary axis one get the following equation

$$\begin{aligned} U^a_{WK}(x) &= \frac{2\alpha}{\pi}\sum_{\kappa=\pm 1}^{\pm\infty} \mid \kappa \mid \int_0^\infty d\omega \int_0^\infty dyy^2 \int_0^\infty dzz^2 \frac{1}{max(x,y)} V(z) \\ &\times \sum_{i,k=1}^{2} Re(F^{ik}_\kappa(i\omega,y,z)(G^{ik}_\kappa(i\omega,y,z) - F^{ik}_\kappa(i\omega,y,z))), \quad (8) \end{aligned}$$

where G^{ik}_κ are the radial components of the partial contributions to the bound-electron Green function and F^{ik}_κ are the radial components of the partial contributions to the free-electron Green function. The homogeneously charged spherical shell model of the nucleus was used in the calculation of U^a_{WK}. In this case the Green function is expressed analytically in terms of the Whittaker, Bessel, and Hankel functions [10, 3]. The reduced Green function which appears in (1) as well as the $1s$ state wave function were calculated for the Fermi model. The κ series in (8) converges rapidly with κ increasing, so we needed $|\kappa| \leq 5$ to reach the relative accuracy not worse than 10^{-4}.

3.2 Diagrams "b"

The calculation of ΔE_b was carried out in the same way as ΔE_a. The corresponding expression for the Uehling operator is also well known:

$$U^b_{Uehl}(\mathbf{x},\mathbf{y}) = \alpha\frac{\alpha_{1\mu}\alpha_2^\mu}{\mid \mathbf{x}-\mathbf{y} \mid}\frac{2\alpha}{3\pi}\int_1^\infty dt(1+\frac{1}{2t^2})\frac{\sqrt{t^2-1}}{t^2}\exp(-2m \mid \mathbf{x}-\mathbf{y} \mid t)\,. \quad (9)$$

As before, the calculation of this term was made using the Fermi model of the nucleus charge distribution. The uncertainty of the results was estimated in the same way as for the Uehling part of the diagrams "a".

The calculation of the Wichman-Kroll contribution of the diagram "b" was done by summing the partial differences between the expression (5) and

the corresponding expression with the bound-electron Green functions replaced by the free-electron Green functions. However, we encountered some obstacles trying to implement this scheme directly. The origin of these obstacles is that, in the direct numerical calculation, it is difficult to reach a full cancelation of some terms with large magnitude. To solve this problem we used the Furry theorem. Taking into account that according to this theorem only terms containing even powers of Z give nonzero contributions to the diagram shown in Fig 1b, the Coulomb Green function was divided into two parts, each containing only odd or even powers of the nucleus charge. After that the expression $Tr(G_k G_k)$ was replaced with $Tr(G_k^{odd} G_k^{odd} + G_k^{even} G_k^{even})$. Then the last trace was calculated analytically (see Appendix C). To control the accuracy of this procedure we calculated the correction due to interaction with an additional external field $\Delta V = -\frac{\alpha}{r}$ to the first order vacuum polarization contribution for a point nucleus (Fig.2). In this case the sum of all the corrections must be equal to the value dE_{VP}/dZ where E_{VP} is the first order vacuum polarization contribution. The results of the test are given in the table 1. The calculation of the Wichman-Kroll contribution of the diagram shown in Fig.1b was carried out for a point nucleus. However, because of the smallness of this contribution ($\sim$ 0.002 eV for Z=90), the finite nuclear size correction can be neglected.

The complete results of the calculation are presented in the table 2. The values of the root-mean-square charge radii were taken from [13]. For comparison, the results of a previous calculation done in [8] are listed in the last column of the table.

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A

The spurious gauge dependent piece of the light-by-light scattering contribution for the diagram shown in Fig.1b is given by

$$\begin{aligned}
\Delta E_M &= e^4 \sum_P (-1)^P \xi_M \,, && (10) \\
\xi_M &= \frac{i}{2\pi} \int_{C_F} d\omega \int d\mathbf{x}_1 d\mathbf{x}_2 A_\mu^{(a)}(\mathbf{x}_1) A_\nu^{(b)}(\mathbf{x}_2) \int d\mathbf{y}_1 d\mathbf{y}_2 \\
&\times Tr[\alpha^\mu F_M(\omega, \mathbf{x}_1, \mathbf{y}_1) V(\mathbf{y}_1) F_M(\omega, \mathbf{y}_1, \mathbf{x}_2) \\
&\times \alpha^\nu F_M(\omega, \mathbf{x}_2, \mathbf{y}_2) V(\mathbf{y}_2) F_M(\omega, \mathbf{y}_2, \mathbf{x}_1) \\
&+ \alpha^\mu F_M(\omega, \mathbf{x}_1, \mathbf{x}_2) \alpha^\nu F_M(\omega, \mathbf{x}_2, \mathbf{y}_1) \\
&\times V(\mathbf{y}_1) F_M(\omega, \mathbf{y}_1, \mathbf{y}_2) V(\mathbf{y}_2) F_M(\omega, \mathbf{y}_2, \mathbf{x}_1) \\
&+ \alpha^\mu F_M(\omega, \mathbf{x}_1, \mathbf{y}_1) V(\mathbf{y}_1) F_M(\omega, \mathbf{y}_1, \mathbf{y}_2) \\
&\times V(\mathbf{y}_2) F_M(\omega, \mathbf{y}_2, \mathbf{x}_2) \alpha^\nu F_M(\omega, \mathbf{x}_2, \mathbf{x}_1)]. && (11)
\end{aligned}$$

where $A_\mu^{(a)}(\mathbf{x}) = \int d\mathbf{y} \psi_{Pa}^\dagger(\mathbf{y}) \alpha^\nu D_{\nu\mu}(0, \mathbf{x}-\mathbf{y}) \psi_a(\mathbf{y})$, $D_{\nu\mu}(\omega, \mathbf{x}-\mathbf{y})$ is the photon propagator, $F_M(\omega, \mathbf{x}, \mathbf{y})$ is the free electron propagator with the electron mass replaced by a hypothetical heavy mass M. Calculating a limit as M becomes infinite yields

$$\begin{aligned}
\lim_{M\to\infty} \xi_M &= \xi^C + \xi^B \,, \\
\xi^C &= \frac{1}{4\pi i} \int_{C_F} d\omega \int d\mathbf{x}_1 A_0^{(a)}(\mathbf{x}_1) A_0^{(b)}(\mathbf{x}_1) V^2(\mathbf{x}_1) \lim_{\mathbf{x}_2 \to \mathbf{x}_1} \\
&\times Tr\Big[\frac{d^3}{d\omega^3} F(\omega, \mathbf{x}_1, \mathbf{x}_2) \Big] \,, && (12) \\
\xi^B &= \frac{i}{4\pi} \int_{C_F} d\omega \int d\mathbf{x}_1 A_i^{(a)}(\mathbf{x}_1) A_k^{(b)}(\mathbf{x}_1) V^2(\mathbf{x}_1) \lim_{\mathbf{x}_2 \to \mathbf{x}_1} \int d\mathbf{y} \\
&\times Tr\Big[\alpha^i \frac{d^2}{d\omega^2} F(\omega, \mathbf{x}_1, \mathbf{y}) \alpha^k F(\omega, \mathbf{y}, \mathbf{x}_2) \Big] \,, && (13)
\end{aligned}$$

where $F(\omega, \mathbf{x}, \mathbf{y})$ is the free-electron propagator; $i, k = 1, 2, 3$. The term ξ^C can be treated in the same way as the light-by-light scattering graph in the third-order vacuum polarization [10, 3]. So, we restrict our consideration to the term ξ^B. Using an identity $\alpha^k = i[H_F, x^k]$, where H_F is the free-electron Hamiltonian, we find

$$\xi^B = \frac{1}{4\pi} \int_{C_F} d\omega \int d\mathbf{x}_1 A_i^{(a)}(\mathbf{x}_1) A_k^{(b)}(\mathbf{x}_1) V^2(\mathbf{x}_1) \lim_{\mathbf{x}_2 \to \mathbf{x}_1} (x_1^k - x_2^k)$$

$$
\begin{aligned}
& \times Tr(\alpha^i \frac{d^2}{d\omega^2} F(\omega, \mathbf{x}_1, \mathbf{x}_2)) \\
= \; & \frac{1}{4\pi} \int d\mathbf{x}_1 A_i^{(a)}(\mathbf{x}_1) A_k^{(b)}(\mathbf{x}_1) V^2(\mathbf{x}_1) \lim_{\Omega\to\infty} \lim_{\mathbf{x}_2\to\mathbf{x}_1} (x_1^k - x_2^k) \\
& \times Tr(\alpha^i \frac{d}{d\omega} F(\omega, \mathbf{x}_1, \mathbf{x}_2)) \Big|_{\omega=-i\Omega}^{\omega=i\Omega} .
\end{aligned} \tag{14}
$$

The first derivative of F with respect to ω is

$$
\frac{dF}{d\omega} = -\Big[\frac{d}{x} i\boldsymbol{\alpha}\cdot\mathbf{x} + \beta m + \omega + \frac{d}{\omega x}\Big] \frac{\exp{(-dx)}}{4\pi} \frac{\omega}{d} \, , \tag{15}
$$

where $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$, $x = |\mathbf{x}|$, $d = \sqrt{m^2 - \omega^2}$. We can see that ξ^B is equal to zero independently of the order of taking limits in (A5). In the partial expansion the terms $F_\kappa^{ik}(\omega, \mathbf{x}, \mathbf{y})$ are finite together with their first derivatives with respect to ω. It follows that the sum of a finite number of terms in the κ series is equal zero.

B

Let us represent the Uehling potential in the form:

$$
U^a_{Uehl}(r) \;=\; -\alpha Z \frac{2\alpha}{3\pi} \int\limits_1^\infty dt \frac{\pi}{mrt} (1 + \frac{1}{2t^2}) \frac{\sqrt{t^2-1}}{t^2} \exp{(-2mrt)} A(r,t) \, , \tag{16}
$$

where

$$
A(r,t) = \int_0^\infty dr' r' \rho(r') [\exp{(-2m(|r-r'|-r)t)} - \exp{(-2mr't)}]. \tag{17}
$$

Expansion of the exponents in the Taylor series yields:

$$
[\exp{(-2m(|r-r'|-r)t)} - \exp{(-2mr't)}] = 4tmr_< + \mathcal{O}(m^2 r' r_< t^2). \tag{18}
$$

Taking into account that

$$
\int_0^\infty dr' \rho(r') r' r_< = -\frac{r}{4\pi\alpha Z} V(r) \tag{19}
$$

one can get

$$
U^a_{Uehl}(r) \approx V(r) \frac{2\alpha}{3\pi} \int\limits_1^\infty dt (1 + \frac{1}{2t^2}) \frac{\sqrt{t^2-1}}{t^2} \exp(-2mrt) \, . \tag{20}
$$

C

The analytical expressions for the radial components of the Coulomb Green function are well known (see ,e.g.,[14]). In the units $\hbar = c = m = 1$ for $x_1 < x_2$ we have ($G = (\omega - H)^{-1}$)

$$
\begin{aligned}
G^{11}_\kappa(\omega, x_1, x_2) &= -(1+\omega)Q\left[(\lambda-\nu)M_{\nu-\frac{1}{2},\lambda}(2dx_1) - (\kappa - \frac{\alpha Z}{d})M_{\nu+\frac{1}{2},\lambda}(2dx_1)\right] \\
&\quad \times \left[(\kappa + \frac{\alpha Z}{d})W_{\nu-\frac{1}{2},\lambda}(2dx_2) + W_{\nu+\frac{1}{2},\lambda}(2dx_2)\right] , \\
G^{12}_\kappa(\omega, x_1, x_2) &= -dQ\left[(\lambda-\nu)M_{\nu-\frac{1}{2},\lambda}(2dx_1) - (\kappa - \frac{\alpha Z}{d})M_{\nu+\frac{1}{2},\lambda}(2dx_1)\right] \\
&\quad \times \left[(\kappa + \frac{\alpha Z}{d})W_{\nu-\frac{1}{2},\lambda}(2dx_2) - W_{\nu+\frac{1}{2},\lambda}(2dx_2)\right] , \\
G^{21}_\kappa(\omega, x_1, x_2) &= -dQ\left[(\lambda-\nu)M_{\nu-\frac{1}{2},\lambda}(2dx_1) + (\kappa - \frac{\alpha Z}{d})M_{\nu+\frac{1}{2},\lambda}(2dx_1)\right] , \\
&\quad \times \left[(\kappa + \frac{\alpha Z}{d})W_{\nu-\frac{1}{2},\lambda}(2dx_2) + W_{\nu+\frac{1}{2},\lambda}(2dx_2)\right] , \\
G^{22}_\kappa(\omega, x_1, x_2) &= -(1-\omega)Q\left[(\lambda-\nu)M_{\nu-\frac{1}{2},\lambda}(2dx_1) + (\kappa - \frac{\alpha Z}{d})M_{\nu+\frac{1}{2},\lambda}(2dx_1)\right] \\
&\quad \times \left[(\kappa + \frac{\alpha Z}{d})W_{\nu-\frac{1}{2},\lambda}(2dx_2) - W_{\nu+\frac{1}{2},\lambda}(2dx_2)\right] ,
\end{aligned}
\tag{21}
$$

where $d = \sqrt{1-\omega^2}$, $\lambda = (\kappa^2 - (\alpha Z)^2)$, $\nu = \frac{\alpha Z \omega}{d}$, $Q = \frac{1}{4d^2(x_1 x_2)^{\frac{3}{2}}} \frac{\Gamma(\lambda-\nu)}{\Gamma(1+2\lambda)}$, $M_{\alpha,\beta}$, and $W_{\alpha,\beta}$ are the Whittaker functions. For $x_1 > x_2$ the radial Green functions can be obtained from the symmetry condition

$$
G^{ik}_\kappa(\omega, x_1, x_2) = G^{ki}_\kappa(\omega, x_2, x_1) \,.
$$

Defining

$$
\begin{aligned}
A &= -Q(\lambda-\nu)M_{\nu-\frac{1}{2},\lambda}(2dx_1)W_{\nu-\frac{1}{2},\lambda}(2dx_2) \,, \\
B &= -Q(\lambda-\nu)M_{\nu-\frac{1}{2},\lambda}(2dx_1)W_{\nu+\frac{1}{2},\lambda}(2dx_1) \,, \\
C &= -QM_{\nu+\frac{1}{2},\lambda}(2dx_1)W_{\nu-\frac{1}{2},\lambda}(2dx_2) \,, \\
D &= -QM_{\nu+\frac{1}{2},\lambda}(2dx_1)W_{\nu+\frac{1}{2},\lambda}(2dx_2) \,,
\end{aligned}
\tag{22}
$$

and taking into account that for $\omega = i\varepsilon$, where ε is real, changing the sign of Z is equal to replacing A, B, C, and D with their complex conjugated

values, we find for the radial parts of the Coulomb Green function containing only odd or even powers of Z the following expressions

$$\begin{aligned}
G_\kappa^{11,odd} &= (1+\omega)\{i(\kappa Im(A-D)+Im(B)-\gamma Im(C)) \\
&\quad +\frac{\alpha Z}{d}Re(A+D)\}\,, \\
G_\kappa^{12,odd} &= d\{i(\kappa Im(A+D)-Im(B)-\gamma Im(C))+\frac{\alpha Z}{d}Re(A-D)\}\,, \\
G_\kappa^{21,odd} &= d\{i(\kappa Im(A+D)+Im(B)+\gamma Im(C))+\frac{\alpha Z}{d}Re(A-D)\}\,, \\
G_\kappa^{22,odd} &= (1-\omega)\{i(\kappa Im(A-D)-Im(B)+\gamma Im(C)) \\
&\quad +\frac{\alpha Z}{d}Re(A+D)\}\,, \qquad (23) \\
G_\kappa^{11,even} &= (1+\omega)\{\kappa Re(A-D)+Re(B)-\gamma Re(C) \\
&\quad +i\frac{\alpha Z}{d}Im(A+D)\}\,, \\
G_\kappa^{12,even} &= d\{\kappa Re(A+D)-Re(B)-\gamma Re(C)+i\frac{\alpha Z}{d}Im(A-D)\}\,, \\
G_\kappa^{21,even} &= d\{\kappa Re(A+D)+Re(B)+\gamma Re(C)+i\frac{\alpha Z}{d}Im(A-D)\}\,, \\
G_\kappa^{22,even} &= (1-\omega)\{\kappa Re(A-D)-Re(B)+\gamma Re(C) \\
&\quad +i\frac{\alpha Z}{d}Im(A+D)\}\,, \qquad (24)
\end{aligned}$$

where $\gamma=\kappa^2-\frac{(\alpha Z)^2}{d^2}$.

In the calculation of the Wichman-Kroll part of the diagram shown in Fig.1b the expression $Re(Tr(\alpha^\nu G(\omega,x,y)\alpha^\mu G(\omega,y,x)))$ appears. After having integrated over the angles we have to evaluate the following expressions

$$S_0 = Re(\sum_{sign(\kappa)=-1}^{1}\sum_{i,k=1}^{2}(G_\kappa^{ik}(\omega,x,y))^2)\,, \qquad (25)$$

$$S_1 = Re(\sum_{sign(\kappa)=-1}^{1}(G_\kappa^{11}G_\kappa^{22}+G_\kappa^{22}G_\kappa^{11}+G_\kappa^{12}G_\kappa^{21}+G_\kappa^{21}G_\kappa^{12})) \qquad (26)$$

$$S_2 = 2Re(\sum_{sign(\kappa)=-1}^{1}(G_\kappa^{11}G_{\kappa'}^{22}+G_\kappa^{22}G_{\kappa'}^{11}+G_\kappa^{12}G_{\kappa'}^{21}+G_\kappa^{21}G_{\kappa'}^{12}))\,, \qquad (27)$$

where $\kappa' = -sign(\kappa)(|\kappa| + 1)$. Using (C3)-(C4) and the Furry theorem we transformed these expressions to the following ones

$$\begin{aligned}
S_0 &= 8\{(\frac{(\alpha Z)^2}{d^2} + \kappa^2)(Re(A)^2 + Re(D)^2 - Im(A)^2 - Im(D)^2) \\
&\quad + Re(B)^2 - Im(B)^2 + \gamma(Re(C)^2 - Im(C)^2)\} \\
&\quad - 16\varepsilon\frac{(\alpha Z)}{d}\{Re(A + D)(Im(B) - \gamma Im(C)) \\
&\quad + Im(A + D)(Re(B) - \gamma Re(C))\} \\
&\quad - 16\varepsilon^2\{(\frac{(\alpha Z)^2}{d^2} + \kappa^2)(Re(A)Re(D) + Im(A)Im(D)) \\
&\quad + \gamma(Re(B)Re(C) + Im(B)Im(C))\} \qquad (28) \\
S_1 &= 8d^2\{(\kappa^2 + \frac{(\alpha Z)^2}{d^2})(Re(A)^2 + Re(D)^2 - Im(A)^2 - Im(D)^2) \\
&\quad + \gamma(Im(C)^2 - Re(C)^2) + Im(B)^2 - Re(B)^2\} \qquad (29) \\
S_2 &= 16d^2\{(|\kappa|(|\kappa| + 1) - \frac{(\alpha Z)^2}{d^2}) \\
&\quad \times(Im(A)Im(A') + Im(D)Im(D') - Re(A)Re(A') - Re(D)Re(D')) \\
&\quad + Im(B)Im(B') - Re(B)Re(B') + \gamma\gamma'(Im(C)Im(C') \\
&\quad - Re(C)Re(C'))\} \qquad (30)
\end{aligned}$$

The equations (28)-(30) were used in the numerical calculation.

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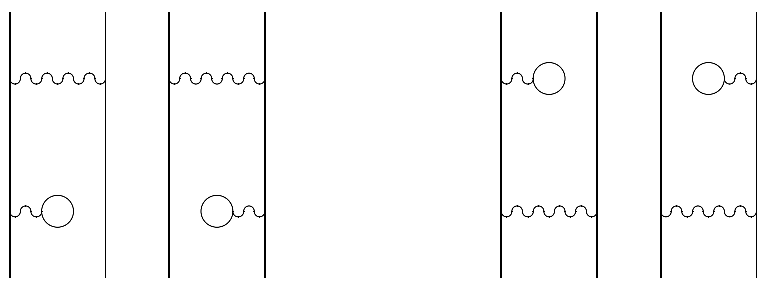
Table 1: Control calculation of the correction due to interaction with an additional external field $\Delta V(r) = -\alpha/r$ to the first order vacuum polarization contribution for the $1s$ state (see Fig.2) (in eV). The label "a" corresponds to the sum of the first two diagrams in Fig 2. The label "b" corresponds to the third diagram in Fig 2. The calculation is carried out for a point nucleus.

Z	ΔE^a_{Uehl}	ΔE^b_{Uehl}	ΔE^a_{WK}	ΔE^b_{WK}	ΔE_{VP}	$\frac{dE_{VP}}{dZ}$
20	-0.019183	-0.006487	0.000072	0.000076	0.025522	0.025521
40	-0.16181	-0.05215	0.00206	0.00221	0.20970	0.20969
60	-0.67136	-0.19446	0.01650	0.01702	0.83231	0.83234
90	-4.5280	-0.9586	0.1970	0.1711	5.1185	5.1186
100	-9.2613	-1.6460	0.4532	0.3543	10.1000	10.0999

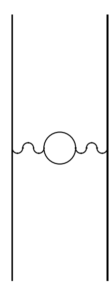
Table 2: Contribution of the vacuum polarization screening diagrams to the ground state energy of two-electron ions (in eV).

Z	$< r^2 >^{1/2}$ (fm)	ΔE^a_{Uehl}	ΔE^b_{Uehl}	ΔE^a_{WK}	ΔE^b_{WK}	ΔE_{VP}	Ref. [8]
20	3.478	0.0090	0.0010	-0.0000	-0.0000	0.0100	
30	3.928	0.0316	0.0035	-0.0003	-0.0000	0.0348	
32	4.072	0.0388	0.0043	-0.0004	-0.0000	0.0427	0.0
40	4.270	0.0807	0.0091	-0.0011	-0.0000	0.0887	
50	4.655	0.176	0.020	-0.003	-0.000	0.192	
54	4.787	0.234	0.027	-0.005	-0.000	0.255	0.2
60	4.914	0.350	0.040	-0.009	-0.000	0.380	
66	5.224	0.515	0.058	-0.016	-0.000	0.557	0.6
70	5.317	0.661	0.074	-0.022	-0.000	0.713	
74	5.373	0.845	0.093	-0.030	-0.000	0.908	0.9
80	5.467	1.215	0.132	-0.049	0.000	1.298	
83	5.533	1.455	0.156	-0.062	0.001	1.550	1.6
90	5.645	2.214(1)	0.230	-0.105	0.002	2.341(1)	
92	5.860	2.493(2)	0.256	-0.122	0.003	2.630(2)	2.6
100	5.886	4.063(4)	0.398	-0.221	0.008	4.248(4)	

Figure 1: Vacuum polarization screening diagrams



a



b

Figure 2: Diagram equation for control calculation of the correction due to interaction with an external field $\Delta V = -\alpha/r$ to the first order vacuum polarization contribution.